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NOTE ON THE BANACH'S MATCH-BOX PROBLEM

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1

A certain mathematician* always carries two match boxes, each of which contains initially N matches. Every time he wants a match, he selects a box at random.

Inevitably a moment occurs when, for the first time, he finds a box empty. At that moment the other box may contain $r = 0, 1, 2, \ldots, N$ matches. We wish now to find the corresponding probabilities.

Let u_r be the probability of the event "when the first box is found empty, the second box contains r matches". This means that out of the first $N + (N - r)$ matches drawn, N are from the first one.

Therefore u_r is the probability of N successes in $2N - r$ trials, each of which has the probability 1/2, so that

$$u_r = \binom{2N-r}{N} \frac{1}{2^{2N-r}}, \qquad r = 0, 1, 2, \ldots, N. \tag{1}$$

Feller calculates the mean value

$$\sum_{r=1}^{N} r u_r = \mu. \tag{2}$$

While he is not able** to calculate this expectation in a direct way, he proceeds in an indirect way, namely using the fact that all the probabilities u_r add to unity

$$\sum_{r=0}^{N} u_r = 1. \tag{3}$$

He says, this relation is not easily verified.

The present paper contains two proofs of the last equation. The first proof (section 2.) uses the method of mathematical induction. The second proof (section 3.) is based on an elementary analytical method. Using the last method we derive in the section 4. the value of μ.

* See [1] p. 108.

** loc. cit. p. 176.

2

With the substitution

$$N - r = k$$

we see that the equation (3)

$$\sum_{r=0}^{N} u_r = \sum_{r=0}^{N} \binom{2N-r}{N} \frac{1}{2^{2N-r}} = 1$$

may be put into the form

$$\sum_{k=0}^{N} \binom{N+k}{k} \frac{1}{2^k} = 2^N. \tag{4}$$

It is evident that this relation holds for $N = 1$ and we will show that if (4) holds for N, then this equation holds also for $N + 1$.

But if we replace N by $N + 1$ in (4), we have

$$\sum_{k=0}^{N+1} \binom{N+1+k}{k} \frac{1}{2^k} = 2^{N+1}$$

or

$$S_{N+1} = 2S_N, \tag{5}$$

where

$$S_N = \sum_{k=0}^{N} \binom{N+k}{k} \frac{1}{2^k}.$$

Now making use of the intentity

$$\binom{N+k}{k-1} + \binom{N+k}{k} = \binom{N+1+k}{k}$$

we obtain

$$S_{N+1} = \sum_{k=0}^{N+1} \binom{N+1+k}{k} \frac{1}{2^k} =$$

$$= \sum_{k=1}^{N+1} \binom{N+k}{k-1} \frac{1}{2^k} + \sum_{k=0}^{N+1} \binom{N+k}{k} \frac{1}{2^k} \tag{6}$$

and so it is only necessary to find the values of both the sums on the right side of this equation.

But after some small modifications we have: first

$$\sum_{k=1}^{N+1} \binom{N+k}{k-1} \frac{1}{2^k} = \frac{1}{2} \sum_{k=0}^{N} \binom{N+1+k}{k} \frac{1}{2^k} =$$

$$= \frac{1}{2} \left[S_{N+1} - \binom{2N+2}{N+1} \frac{1}{2^{N+1}} \right] =$$

$$= \frac{1}{2} S_{N+1} - \frac{1}{2} \frac{2N+2}{N+1} \binom{2N+1}{N} \frac{1}{2^{N+1}} =$$

$$= \frac{1}{2} S_{N+1} - \binom{2N+1}{N} \frac{1}{2^{N+1}};$$

the second sum gives

$$\sum_{k=0}^{N+1}\binom{N+k}{k}\frac{1}{2^k}=S_N+\binom{2N+1}{N+1}\frac{1}{2^{N+1}}$$

so that by substituing both the values in (6) we get the required relation (5).

The assertion (4) is thus established.

3

We introduce for abbreviation the notation

$$(n)_k = n(n-1)\dots(n-k+1)$$

so that the binomial coefficient $\binom{n}{k}$ may be written in the form

$$\binom{n}{k}=\frac{(n)_k}{k!}$$

and for the k-th derivative of x^n we have

$$(x^n)^{(k)} = (n)_k x^{n-k}.$$

Now we go out from the expansion

$$\frac{x^n}{1-x}=\sum_{k=0}^{\infty} x^{n+k}$$

which holds for $|x|<1$. If we differentiate both sides of this equation N times, we obtain the relation

$$\frac{1}{N!}\left(\frac{x^n}{1-x}\right)^{(N)}=\sum_{k=0}^{\infty}\binom{n+k}{N}x^{n-N+k} \tag{7}$$

The derivative on the left side may be easily computed by use of the Leibniz's theorem. In effect, with respect to

$$\left(\frac{1}{1+x}\right)^{(k)}=\frac{k!}{(1-x)^{k+1}}$$

we have succesivly

$$\frac{1}{N!}\left(\frac{x^n}{1-x}\right)^{(N)}=$$
$$=\frac{1}{N!}\sum_{v=0}^{N}\binom{N}{v}(x^n)^{(v)}\left(\frac{1}{1-x}\right)^{(N-v)}=$$
$$=\frac{1}{N!}\sum_{v=0}^{N}\binom{N}{v}(n)_v(N-v)!\frac{x^{n-v}}{(1-x)^{N+1-v}}=$$
$$=\sum_{v=0}^{N}\binom{n}{v}\frac{x^{n-v}}{(1-x)^{N+1-v}}.$$

The value of this derivative for $x = 1/2$ is

$$\left\{\frac{1}{N!}\left(\frac{x^n}{1-x}\right)^{(N)}\right\}_{x=\frac12}=2^{N+1-n}\sum_{v=0}^{N}\binom{n}{v}.$$

Two cases are important for the following: namely $n = N$ and $n = 2N + 1$. Using the identity

$$\sum_{v=0}^{N}\binom{N}{v}=2^N \tag{8}$$

we have for $n = N$

$$\left\{\frac{1}{N!}\left(\frac{x^N}{1-x}\right)^{(N)}\right\}_{x=\frac12}=2\sum_{v=0}^{N}\binom{N}{v}=2^{N+1}. \tag{9}$$

For $n = 2N + 1$ follows

$$\left\{\frac{1}{N!}\left(\frac{x^{2N+1}}{1-x}\right)^{(N)}\right\}_{x=\frac12}=\frac{1}{2^N}\sum_{v=0}^{N}\binom{2N+1}{v} \tag{10}$$

and there remains to calculate the sum on the right side of this equation.

Replacing N by $2N + 1$ in the identity (8) we have

$$2^{2N+1}=\sum_{v=0}^{2N+1}\binom{2N+1}{v}=$$
$$=\sum_{v=0}^{N}\binom{2N+1}{v}+\sum_{v=N+1}^{2N+1}\binom{2N+1}{v}=$$
$$=\sum_{v=0}^{N}\binom{2N+1}{v}+\sum_{v=N+1}^{2N+1}\binom{2N+1}{2N+1-v}=$$
$$=\sum_{v=0}^{N}\binom{2N+1}{v}+\sum_{v=0}^{N}\binom{2N+1}{v}=$$
$$=2\sum_{v=0}^{N}\binom{2N+1}{v}$$

so that

$$\sum_{v=0}^{N} \binom{2N+1}{v} = 2^{2N}.$$

Substituing this value in (10) we obtain

$$\left\{\frac{1}{N!}\left(\frac{x^{2N+1}}{1-x}\right)^{(N)}\right\}_{x=\frac{1}{2}} = 2^N.$$

Now turning back to the equation (7) we have the following relations

$$\sum_{k=0}^{\infty} \binom{N+k}{N} \frac{1}{2^k} = 2^{N+1},$$

$$\sum_{k=N+1}^{\infty} \binom{N+k}{N} \frac{1}{2^k} =$$

$$= \sum_{k=0}^{\infty} \binom{2N+1+k}{N} \frac{1}{2^{N+1+k}} = 2^N$$

and with these results we obtain finally the required relation (4). Namely

$$\sum_{k=0}^{N} \binom{N+k}{N} \frac{1}{2^k} =$$

$$= \sum_{k=0}^{\infty} \binom{N+k}{N} \frac{1}{2^k} - \sum_{k=N+1}^{\infty} \binom{N+k}{N} \frac{1}{2^k} =$$

$$= 2^{N+1} - 2^N = 2^N.$$

It is necessary to remark that we can obtain the equation (9) more easily as follows. As

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

we have for $|x| < 1$ the expansion

$$(1-x)^{-(N+1)} = \sum_{k=0}^{\infty} \binom{-N-1}{k} (-x)^k =$$

$$= \sum_{k=0}^{\infty} \binom{N+k}{k} x^k$$

and from this for $x = 1/2$ we find (9).

4

By the same method as we have used in the preceding section to proof the relation (4), we can calculate the mean value (2).

There is

$$\mu = \sum_{r=1}^{N} r u_r = \sum_{r=1}^{N} r \binom{2N-r}{N} \frac{1}{2^{2N-r}} =$$

$$= \sum_{k=0}^{N-1} (N-k) \binom{N+k}{N} \frac{1}{2^{N+k}} =$$

$$= \frac{N}{2^N} \sum_{k=0}^{N} \binom{N+k}{N} \frac{1}{2^k} - \frac{1}{2^N} \sum_{k=1}^{N} \binom{N+k}{N} \frac{k}{2^k}$$

or with respect to (3)

$$\mu = N - \frac{1}{2^N} \sum_{k=1}^{N} \binom{N+k}{N} \frac{k}{2^k} \tag{11}$$

so that it remains to evaluate the second term of this expression.

For this purpose we derivate once more the equation (7). Multiplying the resulting equation by x we obtain

$$\frac{x}{N!}\left(\frac{x^n}{1-x}\right)^{(N+1)} =$$

$$= \sum_{k=0}^{\infty} \binom{n+k}{N} (n-N+k) x^{n-N+k}. \tag{12}$$

The derivative on the left side may be calculated again by use of the Leibniz's theorem. We obtain

$$\frac{x}{N!}\left(\frac{x^n}{1-x}\right)^{(N+1)} =$$

$$= \frac{x}{N!} \sum_{v=0}^{N+1} \binom{N+1}{v} (x^n)^{(v)} \left(\frac{1}{1-x}\right)^{(N+1-v)} =$$

$$= \frac{x}{N!} \sum_{v=0}^{N+1} \binom{N+1}{v} (n)_v (N+1-v)! \frac{x^{n-v}}{(1-x)^{N+2-v}} =$$

$$= (N+1) \sum_{v=0}^{N+1} \binom{n}{v} \frac{x^{n+1-v}}{(1-x)^{N+2-v}}.$$

In the following we shall need the values of this derivative for $x = 1/2$ and $n = N$, $n = 2N + 1$.

For $n = N$ and $x = 1/2$ we get

$$\left\{\frac{x}{N!}\left(\frac{x^N}{1-x}\right)^{(N+1)}\right\}_{x=\frac{1}{2}} =$$

$$= 2(N+1)\sum_{v=0}^{N}\binom{N}{v} = (N+1)\,2^{N+1}.$$

For $n = 2N + 1$ and $x = 1/2$ there is

$$\left\{\frac{x}{N!}\left(\frac{x^{2N+1}}{1-x}\right)^{(N+1)}\right\}_{x=\frac{1}{2}} =$$

$$= \frac{N+1}{2^N}\sum_{v=0}^{N+1}\binom{2N+1}{v} = \frac{N+1}{2^N}\left[2^{2N} + \binom{2N+1}{N+1}\right] =$$

$$= (N+1)\,2^N + \frac{N+1}{2^N}\binom{2N+1}{N}.$$

Now if we return to the equation (12) we have first as result the relation

$$\sum_{k=1}^{\infty}\binom{N+k}{N}\frac{k}{2^k} = (N+1)\,2^{N+1}. \qquad (13)$$

Further it follows

$$(N+1)\,2^N + \frac{N+1}{2^N}\binom{2N+1}{N} =$$

$$= \sum_{k=0}^{\infty}\binom{2N+1+k}{N}\frac{N+1+k}{2^{N+1+k}} = \sum_{k=N+1}^{\infty}\binom{N+k}{N}\frac{k}{2^k}.$$

On base of these results we can now calculate μ. There is

$$\mu = N - \frac{1}{2^N}\left[\sum_{k=1}^{\infty}\binom{N+k}{N}\frac{k}{2^k} - \sum_{k=N+1}^{\infty}\binom{N+k}{N}\frac{k}{2^k}\right] =$$

$$= N - \frac{1}{2^N}\left[(N+1)\,2^{N+1} - (N+1)\,2^N - \frac{N+1}{2^N}\binom{2N+1}{N}\right] =$$

$$= \frac{N+1}{2^{2N}}\binom{2N+1}{N} - 1 = \frac{2N+1}{2^{2N}}\binom{2N}{N} - 1.$$

Let us only remarque that the equation (13) can be obtained in a shorter way from the last equation of the section 3. Derivative and multiplication by x gives

$$x(N+1)(1-x)^{-(N+1)-1} = \sum_{k=1}^{\infty}\binom{N+k}{k}kx^k$$

therefrom we obtain for $x = 1/2$ the equation (13).

REFERENCES

[1] Feller W., *An Introduction to Probability Theory and its Applications* I, New York 1950.

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O JISTÉ BANACHOVĚ ÚLOZE Z POČTU PRAVDĚPODOBNOSTI

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Výtah

V podstatě jde o tuto úlohu ([1], str. 108, 176). Máme dvě krabice, z nichž každá obsahuje N předmětů. Předměty v obou krabicích jsou stejné. Potřebujeme-li předmět, zvolíme nejprve úplně libovolně jednu z obou krabic a z ní teprve předmět vyjmeme.

Označme u_r pravděpodobnost tohoto jevu „jedna krabice je prázdná a druhá obsahuje r předmětů", přičemž $r = 0, 1, 2, \ldots, N$. Tento případ po určitém počtu „tahů" jistě nastane. Snadno se zjistí, že je

$$u_r = \binom{2N-r}{N}\frac{1}{2^{2N-r}}, \qquad r = 0, 1, 2, \ldots, N. \qquad (1)$$

Feller počítá střední hodnotu

$$\mu = \sum_{r=1}^{N} ru_r \qquad (2)$$

a protože, jak sám říká, není s to počítat μ přímo, postupuje nepřímo užívaje té okolnosti, že součet všech pravděpodobností u_r dává jedničku

$$\sum_{r=0}^{N} u_r = 1. \qquad (3)$$

O této rovnici Feller říká, že ji nelze snadno dokázat.

Předcházející práce obsahuje dva důkazy vztahu (3). A sice ve 2. odstavci je podán důkaz úplnou indukcí, ve 3. odstavci je tato rovnice dokázána jednoduchou analytickou metodou, při níž se vychází z vytvořující funkce pro binomické koeficienty $\binom{n+k}{k}$.

V posledním odstavci je této metody užito k výpočtu střední hodnoty μ.